QP CODE: 21000686A

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Reg No : .....

Name : .....

M Sc DEGREE (CSS) EXAMINATION, JULY 2021

Fourth Semester

Faculty of Science

CORE - ME010401 - SPECTRAL THEORY

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 Admission Onwards

F208C857

Time: 3 Hours Weightage: 30

Part A (Short Answer Questions)

Answer any **eight** questions.

Weight **1** each.

1. Let X be a normed space and $g_x \in X''$ defined by $g_x(f) = f(x)$. Prove that $\|g_x\| = \|x\|$ for every fixed $x \in X$.
2. Let X and Y are normed spaces and $(T_n) \in B(X,Y)$ such that $T_n \to T$ uniformly. Prove that (T_n) converges to T weakly.
3. Define fixed point of a mapping defined on a set. Give an example of a mapping with two fixed points.
4. Define similar matrices. Prove that similar matrices have the same eigenvalues.
5. Define spectrum of a linear operator $T : D(T) \to X$, where $X \neq \{0\}$ is a complex normed space and $D(T) \subset X$. Also, define the three partitions of the spectrum.
6. Define inverse of an element $x \in A$, where A is an algebra with identity. Show that inverse of an element if it exist is unique.
7. Show that the resolvent $\rho(x)$ is open, where $x \in A$ and A is a complex Banach algebra with identity.
8. Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y and (T_n) is strongly operator convergent to T. Then show by an example that T need not be compact.
9. *Define self-adjoint linear operator on a Hilbert space. Prove that the eigen vectors corresponding to distinct eigen values of a bounded self-adjoint linear operator on a complex Hilbert space are orthogonal.*
10. Define orthogonal projection on a Hilbert space. State and prove any two properties of an orthogonal projection.

(8×1=8 weightage)

Part B (Short Essay/Problems)

Answer any **six** questions.

Weight **2** each.

11. State and prove Uniform Boundedness theorem.
12. Prove that in a normed space X we have $x_n \xrightarrow{w} x$ if and only if:
    (A) The sequence $(\|x_n\|)$ is bounded.
    (B) For every element f of a total subset $M \subset X'$ we have $f(x_n) \to f(x)$.

13. Prove that the eigenvectors $x_1, x_2, \cdots, x_n$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of a linear operator T defined on a vector space, are linearly independant.

14. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

15. If B is a totally bounded subset of a metric space X, prove that B contains a finite ϵ - net for every $\epsilon > 0$.

16. If T is a compact linear operator on a normed space X, then prove that for every $\lambda \neq 0$, X can be represented in the form $X = \mathcal{N}(T_\lambda^r) \oplus T_\lambda^r(X)$.

17. Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Prove that $\|T\| = \sup_{\|x\|=1} |\langle Tx, x \rangle|$.

18. Let P_1 and P_2 be projections defined on a Hilbert space H and let $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$. If $Y_1 \subset Y_2$, prove that the difference $P = P_2 - P_1$ is a projection on H onto $Y = Y_2 \cap Y_1^{\perp}$.

(6×2=12 weightage)

Part C (Essay Type Questions)

Answer any **two** questions.

Weight **5** each.

19. (a) Let X and Y be Banach spaces. Let $T : X \to Y$ be a surjective bounded linear operator. Prove that the image $T(B_0)$ of the open unit ball $B_0 = B(0; 1) \subset X$ contains an open ball about $0 \in Y$.

(b) State and prove Bounded Inverse Theorem.

20. Let $T \in B(X, X)$, where X is a complex Banach space. Then:

(a) Prove that the spectrum of T is closed.

(b) Prove that the spectrum of T is compact.

21. If T is a is a compact linear operator on a normed space X. Prove that for every $\lambda \neq 0$,

a) $\dim \mathcal{N}(T_\lambda^n) < \infty$ and $\{0\} = \mathcal{N}(T_\lambda^0) \subset \mathcal{N}(T_\lambda^0) \subset \mathcal{N}(T_\lambda^2) \subset \cdots$

b) the range of T_λ^n is closed and $X = T_\lambda^0(X) \supset T\lambda(X) \supset T_\lambda^2(X) \supset \cdots$.

22. Let (T_n) be a sequence of bounded self-adjoint linear operators on a complex Hilbert space H such that $T_1 \leq T_2 \leq \cdots \leq T_n \leq \cdots \leq K$ where K is a bounded self-adjoint linear operator on H. Suppose that any T_j commutes with K and with every T_m. Then prove that (T_n) is strongly operator convergent and the limit operator T is linear, bounded and self-adjoint and satisfies $T \leq K$.

(2×5=10 weightage)